SUPPLEMENTAL APPENDIX

A Convenient Omitted Variable Bias Formula for Treatment Effect Models

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A Calculating The Partial Correlation Matrix in a Bivariate Treatment Model

In Section 3.1 of the paper the omitted variable bias for the coefficient on x_1 is given in equation 3. To see this, note that the inverse of $(X_1'X_1)$ is:

$$(X_1'X_1)^{-1} = \begin{bmatrix} \frac{1}{N-N_{x_1}} & -\frac{1}{N-N_{x_1}} \\ -\frac{1}{N-N_{x_1}} & \frac{N}{N_{x_1}(N-N_{x_1})} \end{bmatrix}.$$

Combining this inverse with $X_1'X_2$ gives the partial correlation matrix between the omitted X_2 and included X_1 as:

$$\delta = (X_1'X_1)^{-1}X_1'X_2 = \begin{bmatrix} \frac{1}{N-N_{x_1}} & -\frac{1}{N-N_{x_1}} \\ -\frac{1}{N-N_{x_1}} & \frac{N}{N_{x_1}(N-N_{x_1})} \end{bmatrix} \begin{bmatrix} N_{x_2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{N_{x_2}}{N-N_{x_1}} \\ -\frac{N_{x_2}}{N-N_{x_1}} \end{bmatrix},$$

as documented in section 3.1. Finally, substituting this into equation 2 gives that the expectation on the 2×1 parameter vector $\widehat{\beta}_1^{ovb}$. This is:

$$E[\widehat{\beta}_1^{ovb}|X] = \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2 \tag{A1}$$

$$= \begin{bmatrix} \beta_1^0 \\ \beta_1^1 \end{bmatrix} + \begin{bmatrix} \frac{N_{x_2}}{N-N_{x_1}} \\ -\frac{N_{x_2}}{N-N_{x_1}} \end{bmatrix}\beta_2, \tag{A2}$$

and so the expectation of the estimate on the treatment indicator (the second element of A2) is $\beta_1^1 - \beta_2\frac{N_{x_2}}{N-N_{x_1}}$, as documented in equation 3.

B Inverting the Arrowhead Matrix in the Multivariate Treatment Effects Model

In the generalised treatment effects model with k_1 included treatment indicators and k_2 excluded treatment indicators, the $(X_1'X_1)$ and $(X_1'X_2)$ matrices are given as:

$$(X_1'X_1) = \begin{bmatrix} N & N_{x_1^1} & N_{x_1^2} & \cdots & N_{x_1^{k_1}} \\ N_{x_1^1} & N_{x_1^1} & 0 & \cdots & 0 \\ N_{x_1^2} & 0 & N_{x_1^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{x_1^{k_1}} & 0 & 0 & \cdots & N_{x_1^{k_1}} \end{bmatrix} \qquad (X_1'X_2) = \begin{bmatrix} N_{x_2^1} & N_{x_2^2} & \cdots & N_{x_2^{k_2}} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Here the $(X_1'X_1)$ matrix takes the form of an arrowhead with each element denoting the quantity of observations receiving each particular treatment, or the total number of observations. To calculate the partial correlation matrix between X_2 and X_1, the inverse of the $X_1'X_1$ matrix is found following the known inverse formula for arrowhead matrices. This formula assumes a $J \times J$ arrowhead matrix of the form:

$$A = \begin{bmatrix} \alpha & z' \\ z & D \end{bmatrix}$$

where α is a scalar, z a vector with $J-1$ elements, and D a diagonal $J-1 \times J-1$ matrix with zeros on the off-diagonal. If each of the diagonal terms is non-zero (an assumption which is met by construction provided that each treatment variable has at least one treated observation), the inverse is equal to

$$A^{-1} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & D^{-1} \end{bmatrix} + \rho u u'$$

where $u = [-1 \quad D^{-1}z]'$ and $\rho = \frac{1}{\alpha - z'D^{-1}z}$.

In this case given the structure of $X_1'X_1$, we have that $\alpha = N$, $z = [N_{x_1^1} \quad N_{x_1^2} \quad \cdots \quad N_{x_1^{k_1}}]'$ and D is the matrix contained from row 2 to J and column 2 to J of $(X_1'X_1)$. Putting this together gives the inverse as:

$$(X_1'X_1)^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1/N_{x_1^1} & 0 & \cdots & 0 \\ 0 & 0 & 1/N_{x_1^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/N_{x_1^{k_1}} \end{bmatrix} + \frac{1}{\lambda} \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

where $\lambda = N - N_{x_1^1} - N_{x_1^2} - \cdots - N_{x_1^{k_1}}$, and simplifying gives:

$$(X_1'X_1)^{-1} = \begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \cdots & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{1}{N_{x_1^1}} + \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{N_{x_1^2}} + \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{N_{x_1^{k_1}}} + \frac{1}{\lambda} \end{bmatrix}.$$

Combining this inverse with the matrix $(X_1'X_2)$ gives the matrix of partial correlations as:

$$\delta = \begin{bmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda} & -\frac{1}{\lambda} & \cdots & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{1}{N_{x_1^1}} + \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{N_{x_1^2}} + \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{N_{x_1^{k_1}}} + \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} N_{x_2^1} & N_{x_2^2} & \cdots & N_{x_2^{k_2}} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \frac{N_{x_2^1}}{\lambda} & \frac{N_{x_2^2}}{\lambda} & \cdots & \frac{N_{x_2^{k_2}}}{\lambda} \\ -\frac{N_{x_2^1}}{\lambda} & -\frac{N_{x_2^2}}{\lambda} & \cdots & -\frac{N_{x_2^{k_2}}}{\lambda} \\ -\frac{N_{x_2^1}}{\lambda} & -\frac{N_{x_2^2}}{\lambda} & \cdots & -\frac{N_{x_2^{k_2}}}{\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{N_{x_2^1}}{\lambda} & -\frac{N_{x_2^2}}{\lambda} & \cdots & -\frac{N_{x_2^{k_2}}}{\lambda} \end{bmatrix}$$

as documented in Section 3.2 of the paper. Substituting this into equation 2 gives the OVB formula for each of the k_1 included treatment variables displayed in equation 4.

C Inverting the 2×2 Block Matrix for OVB in Simple Difference-in-Differences

In the difference-in-differences model with one included and one excluded treatment indicator (plus corresponding fixed effects), given that X_{1D} is a Boolean matrix, and various columns are mutually exclusive, we can write $(X_{1D}'X_{1D})$ and $(X_{1D}'X_{2D})$ as:

$$(X_{1D}'X_{1D}) = \begin{bmatrix} N & N_{x_1} & N_{x_{1t}} & N_t \\ N_{x_1} & N_{x_1} & N_{x_{1t}} & N_{x_{1t}} \\ N_{x_{1t}} & N_{x_{1t}} & N_{x_{1t}} & N_{x_{1t}} \\ N_t & N_{x_{1t}} & N_{x_{1t}} & N_t \end{bmatrix} \qquad (X_{1D}'X_{2D}) = X_{1D}'[x_{2i} \quad (x_2 \cdot \text{Post})_{it}] = \begin{bmatrix} N_{x_2} & N_{x_{2t}} \\ 0 & 0 \\ 0 & 0 \\ N_{x_{2t}} & N_{x_{2t}} \end{bmatrix}.$$

In order to calculate the omitted variable bias we must first invert $(X_{1D}'X_{1D})$. This matrix is a symmetric block matrix, and so can be re-written as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A'_{12} & A_{22} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B'_{12} & B_{22} \end{bmatrix}$$

where each of A_{11}, A_{12} and A_{22} are the 2×2 matrices in each corner of $(X'_{1D}X_{1D})$. The formula for the inverse of a 2×2 block matrix, and the algebra corresponding to $(X'_{1D}X_{1D})$ implies that each element of the inverse has the formula given below (see for example Lu and Shiou (2002)):

$$\begin{aligned} B_{11} &= (A_{11} - A_{12}A_{22}^{-1}A'_{12})^{-1} \\ &= \frac{1}{N_{x_{1t}}(N_t - N_{x_{1t}})} \begin{bmatrix} N_{x_{1t}} & -N_{x_{1t}} \\ -N_{x_{1t}} & N_t \end{bmatrix} \end{aligned} \tag{A3}$$

$$\begin{aligned} B_{22} &= (A_{22} - A'_{12}A_{11}^{-1}A_{12})^{-1} \\ &= \frac{2}{N_{x_{1t}}(N_t - N_{x_{1t}})} \begin{bmatrix} N_t & -N_{x_{1t}} \\ -N_{x_{1t}} & N_{x_{1t}} \end{bmatrix} \end{aligned} \tag{A4}$$

$$\begin{aligned} B_{12} &= -A_{11}^{-1}A_{12}(A_{22} - A'_{12}A_{11}^{-1}A_{12})^{-1} \\ &= \begin{bmatrix} \frac{1}{N_t-N_{x_{1t}}} & -\frac{1}{N_t-N_{x_{1t}}} \\ -\frac{N_t}{N_{x_{1t}}(N_t-N_{x_{1t}})} & \frac{1}{N_t-N_{x_{1t}}} \end{bmatrix}. \end{aligned} \tag{A5}$$

$$\begin{aligned} B'_{12} &= -A_{22}^{-1}A'_{12}(A_{11} - A_{12}A_{22}^{-1}A'_{12})^{-1} \\ &= \begin{bmatrix} \frac{1}{N_t-N_{x_{1t}}} & -\frac{N_t}{N_{x_{1t}}(N_t-N_{x_{1t}})} \\ -\frac{1}{N_t-N_{x_{1t}}} & \frac{1}{N_t-N_{x_{1t}}} \end{bmatrix} \end{aligned} \tag{A6}$$

The first line of each expression above is from the inverse formula for 2×2 block matrices, while the second line for each sub-matrix is resolved simply by linear algebra. Putting the four elements of B together gives:

$$A^{-1} = (X'_{1D}X_{1D})^{-1} = \begin{bmatrix} \frac{1}{N_t-N_{x_{1t}}} & -\frac{1}{N_t-N_{x_{1t}}} & \frac{1}{N_t-N_{x_{1t}}} & -\frac{1}{N_t-N_{x_{1t}}} \\ -\frac{1}{N_t-N_{x_{1t}}} & \frac{N_t}{N_{x_{1t}}(N_t-N_{x_{1t}})} & -\frac{N_t}{N_{x_{1t}}(N_t-N_{x_{1t}})} & \frac{1}{N_t-N_{x_{1t}}} \\ \frac{1}{N_t-N_{x_{1t}}} & -\frac{N_t}{N_{x_{1t}}(N_t-N_{x_{1t}})} & \frac{2N_t}{N_{x_{1t}}(N_t-N_{x_{1t}})} & -\frac{2}{N_t-N_{x_{1t}}} \\ -\frac{1}{N_t-N_{x_{1t}}} & \frac{1}{N_t-N_{x_{1t}}} & -\frac{2}{N_t-N_{x_{1t}}} & \frac{2}{N_t-N_{x_{1t}}} \end{bmatrix}.$$

Finally, post-multiplying $(X_{1D}'X_{1D})^{-1}$ by $(X_{1D}'X_{2D})$ results in the matrix δ:

$$\delta = (X_{1D}'X_{1D})^{-1}(X_{1D}'X_{2D}) = \begin{bmatrix} \frac{N_{x_{2t}}}{N_t - N_{x_{1t}}} & 0 \\ -\frac{N_{x_{2t}}}{N_t - N_{x_{1t}}} & 0 \\ 0 & -\frac{N_{x_{2t}}}{N_t - N_{x_{1t}}} \\ 0 & \frac{N_{x_{2t}}}{N_t - N_{x_{1t}}} \end{bmatrix}.$$

as indicated in equation 7 of the paper. To produce the OVB formula on the treatment variable of interest, δ is substituted into equation 6, which results in the solution given in equation 8 for the third element of the β vector (the treatment effect of interest). Given the partial correlation matrix δ above, it is also simple to observe the bias on other (non-treatment) coefficients if desired.

D Inverting the $(X_{1D}'X_{1D})$ Matrix in a Difference-in-Differences Model with Multiple Included and Excluded Treatment Indicators

In the case of a difference-in-differences model with multiple included and multiple excluded treatment indicators (ie $k_1 > 1$ and $k_2 > 1$), we can write the $(X_{1D}'X_{1D})$ (at left) and $(X_{1D}'X_{2D})$ (at right) matrices as:

$$\begin{bmatrix} N & N_{x_1^1} & N_{x_1^2} & \dots & N_{x_1^{k_1}} & N_{x_{1t}^{k_1}} & \dots & N_{x_{1t}^2} & N_{x_{1t}^1} & N_t \\ N_{x_1^1} & N_{x_1^1} & 0 & \dots & 0 & 0 & \dots & 0 & N_{x_{1t}^1} & N_{x_{1t}^1} \\ N_{x_1^2} & 0 & N_{x_1^2} & \dots & 0 & 0 & \dots & N_{x_{1t}^2} & 0 & N_{x_{1t}^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ N_{x_1^{k_1}} & 0 & 0 & \dots & N_{x_1^{k_1}} & N_{x_{1t}^{k_1}} & \dots & 0 & 0 & N_{x_{1t}^{k_1}} \\ N_{x_{1t}^{k_1}} & 0 & 0 & \dots & N_{x_{1t}^{k_1}} & N_{x_{1t}^{k_1}} & \dots & 0 & 0 & N_{x_{1t}^{k_1}} \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N_{x_{1t}^2} & 0 & N_{x_{1t}^2} & \dots & 0 & 0 & \dots & N_{x_{1t}^2} & 0 & N_{x_{1t}^2} \\ N_{x_{1t}^1} & N_{x_{1t}^1} & 0 & \dots & 0 & 0 & \dots & 0 & N_{x_{1t}^1} & N_{x_{1t}^1} \\ N_t & N_{x_{1t}^1} & N_{x_{1t}^2} & \dots & N_{x_{1t}^{k_1}} & N_{x_{1t}^{k_1}} & \dots & N_{x_{1t}^2} & N_{x_{1t}^1} & N_t \end{bmatrix} \quad \begin{bmatrix} N_{x_2^1} & \dots & N_{x_2^{k_2}} & N_{x_{2t}^1} & \dots & N_{x_{2t}^{k_2}} \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ N_{x_{2t}^1} & \dots & N_{x_{2t}^{k_2}} & N_{x_{2t}^1} & \dots & N_{x_{2t}^{k_2}} \end{bmatrix}.$$

Here, determining the inverse of the $(X_{1D}'X_{1D})$ matrix requires inverting a $(2 + 2k_1) \times (2 + 2k_1)$ matrix, based on the k_1 treatment indicators, and identical number of fixed effects, an intercept term, and a time dummy. Given that this is once again a 2×2 block matrix, we can use the identical formula as in section C for the inverse of each block of $(X'_{1D}X_{1D})$. However, now each block is of dimension $(k_1+1) \times (k_1+1)$. Fortunately, given the structure of the difference-in-difference model, each block is an arrowhead matrix, and so the required internal matrices are easily invertible using the arrowhead matrix formula (see Najafi et al. (2014) for discussion, and the formula laid out in section B of this document).

Each element of the inverse is documented below:

$$\begin{aligned}
B_{11} &= (A_{11} - A_{12}A_{22}^{-1}A'_{12})^{-1} = \begin{bmatrix} \frac{1}{\theta_t} & -\frac{1}{\theta_t} & -\frac{1}{\theta_t} & \cdots & -\frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \frac{\theta_t+N_{x^1_{1t}}}{\theta_t N_{x^1_{1t}}} & \frac{1}{\theta_t} & \cdots & \frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \frac{\theta_t+N_{x^2_{1t}}}{\theta_t N_{x^2_{1t}}} & \cdots & \frac{1}{\theta_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \frac{1}{\theta_t} & \cdots & \frac{\theta_t+N_{x^{k_1}_{1t}}}{\theta_t N_{x^{k_1}_{1t}}} \end{bmatrix} \qquad \text{(A7)} \\
\text{where } \theta_t &= N_t - N_{x^1_{1t}} - N_{x^2_{1t}} - \cdots - N_{x^{k_1}_{1t}},
\end{aligned}$$

and

$$\begin{aligned}
B_{22} &= (A_{22} - A'_{12}A_{11}^{-1}A_{12})^{-1} = \begin{bmatrix} \frac{2(\theta_t+N_{x^{k_1}_1})}{\theta_t N_{x^{k_1}_{1t}}} & \cdots & \frac{2}{\theta_t} & \frac{2}{\theta_t} & -\frac{2}{\theta_t} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{2}{\theta_t} & \cdots & \frac{2(\theta_t+N_{x^2_1})}{\theta_t N_{x^2_{1t}}} & \frac{2}{\theta_t} & -\frac{2}{\theta_t} \\ \frac{2}{\theta_t} & \cdots & \frac{2}{\theta_t} & \frac{2(\theta_t+N_{x^1_1})}{\theta_t N_{x^1_{1t}}} & -\frac{2}{\theta_t} \\ -\frac{2}{\theta_t} & \cdots & -\frac{2}{\theta_t} & -\frac{2}{\theta_t} & \frac{2}{\theta_t} \end{bmatrix} \qquad \text{(A8)} \\
\text{where } \theta &= N - N_{x^1_1} - N_{x^1_1} - \cdots - N_{x^{k_1}_1}
\end{aligned}$$

for the diagonal blocks, and

$$B_{12} = -A_{11}^{-1}A_{12}(A_{22} - A'_{12}A_{11}^{-1}A_{12})^{-1} = \begin{bmatrix} \frac{1}{\theta_t} & \cdots & \frac{1}{\theta_t} & \frac{1}{\theta_t} & -\frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \cdots & -\frac{1}{\theta_t} & -\frac{\theta_t+N_{x^1_{1t}}}{\theta_t N_{x^1_{1t}}} & \frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \cdots & -\frac{\theta_t+N_{x^2_{1t}}}{\theta_t N_{x^2_{1t}}} & -\frac{1}{\theta_t} & \frac{1}{\theta_t} \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ -\frac{\theta_t+N_{x^{k_1}_{1t}}}{\theta_t N_{x^{k_1}_{1t}}} & \cdots & -\frac{1}{\theta_t} & -\frac{1}{\theta_t} & \frac{1}{\theta_t} \end{bmatrix}, \qquad \text{(A9)}$$

and

$$B'_{12} = -A_{22}^{-1}A'_{12}(A_{11} - A_{12}A_{22}^{-1}A'_{12})^{-1} = \begin{bmatrix} \frac{1}{\theta_t} & -\frac{1}{\theta_t} & -\frac{1}{\theta_t} & \cdots & -\frac{\theta_t+N_{x_{1t}^{k_1}}}{\theta_t N_{x_{1t}^{k_1}}} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ \frac{1}{\theta_t} & -\frac{1}{\theta_t} & -\frac{\theta_t+N_{x_{1t}^2}}{\theta_t N_{x_{1t}^2}} & \cdots & -\frac{1}{\theta_t} \\ \frac{1}{\theta_t} & -\frac{\theta_t+N_{x_{1t}^1}}{\theta_t N_{x_{1t}^1}} & -\frac{1}{\theta_t} & \cdots & -\frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \frac{1}{\theta_t} & \cdots & \frac{1}{\theta_t} \end{bmatrix} \tag{A10}$$

for the off-diagonal blocks.

Combining A7-A10 gives the following for $(X'_{1D}X_{1D})^{-1}$:

$$(X'_{1D}X_{1D})^{-1} = \begin{bmatrix} \frac{1}{\theta_t} & -\frac{1}{\theta_t} & -\frac{1}{\theta_t} & \cdots & -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \cdots & \frac{1}{\theta_t} & \frac{1}{\theta_t} & -\frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \frac{\theta_t+N_{x_{1t}^1}}{\theta_t N_{x_{1t}^1}} & \frac{1}{\theta_t} & \cdots & \frac{1}{\theta_t} & -\frac{1}{\theta_t} & \cdots & -\frac{1}{\theta_t} & -\frac{\theta_t+N_{x_{1t}^1}}{\theta_t N_{x_{1t}^1}} & \frac{1}{\theta_t} \\ -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \frac{\theta_t+N_{x_{1t}^2}}{\theta_t N_{x_{1t}^2}} & \cdots & \frac{1}{\theta_t} & -\frac{1}{\theta_t} & \cdots & -\frac{\theta_t+N_{x_{1t}^2}}{\theta_t N_{x_{1t}^2}} & -\frac{1}{\theta_t} & \frac{1}{\theta_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \frac{1}{\theta_t} & \cdots & \frac{\theta_t+N_{x_{1t}^{k_1}}}{\theta_t N_{x_{1t}^{k_1}}} & -\frac{\theta_t+N_{x_{1t}^{k_1}}}{\theta_t N_{x_{1t}^{k_1}}} & \cdots & -\frac{1}{\theta_t} & -\frac{1}{\theta_t} & \frac{1}{\theta_t} \\ \frac{1}{\theta_t} & -\frac{1}{\theta_t} & -\frac{1}{\theta_t} & \cdots & -\frac{\theta_t+N_{x_{1t}^{k_1}}}{\theta_t N_{x_{1t}^{k_1}}} & \frac{2(\theta_t+N_{x_1^{k_1}})}{\theta_t N_{x_{1t}^{k_1}}} & \cdots & \frac{2}{\theta_t} & \frac{2}{\theta_t} & -\frac{2}{\theta_t} \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{\theta_t} & -\frac{1}{\theta_t} & -\frac{\theta_t+N_{x_{1t}^2}}{\theta_t N_{x_{1t}^2}} & \cdots & -\frac{1}{\theta_t} & \frac{2}{\theta_t} & \cdots & \frac{2(\theta_t+N_{x_1^2})}{\theta_t N_{x_{1t}^2}} & \frac{2}{\theta_t} & -\frac{2}{\theta_t} \\ \frac{1}{\theta_t} & -\frac{\theta_t+N_{x_{1t}^1}}{\theta_t N_{x_{1t}^1}} & -\frac{1}{\theta_t} & \cdots & -\frac{1}{\theta_t} & \frac{2}{\theta_t} & \cdots & \frac{2}{\theta_t} & \frac{2(\theta_t+N_{x_1^1})}{\theta_t N_{x_{1t}^1}} & -\frac{2}{\theta_t} \\ -\frac{1}{\theta_t} & \frac{1}{\theta_t} & \frac{1}{\theta_t} & \cdots & \frac{1}{\theta_t} & -\frac{2}{\theta_t} & \cdots & -\frac{2}{\theta_t} & -\frac{2}{\theta_t} & \frac{2}{\theta_t} \end{bmatrix}$$

Post-multiplying $(X'_{1D}X_{1D})^{-1}$ by the previous $(X'_{1D}X_{2D})$ matrix results in the partial correlation matrix for the omitted variable bias formula for the difference-in-differences model of interest. This is:

$$\delta = (X'_{1D}X_{1D})^{-1}(X'_{1D}X_{2D}) = \begin{bmatrix} \frac{N_{x_{2t}^1}}{\theta_t} & \dots & \frac{N_{x_{2t}^{k_2}}}{\theta_t} & 0 & \dots & 0 \\ -\frac{N_{x_{2t}^1}}{\theta_t} & \dots & -\frac{N_{x_{2t}^{k_2}}}{\theta_t} & 0 & \dots & 0 \\ -\frac{N_{x_{2t}^1}}{\theta_t} & \dots & -\frac{N_{x_{2t}^{k_2}}}{\theta_t} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{N_{x_{2t}^1}}{\theta_t} & \dots & -\frac{N_{x_{2t}^{k_2}}}{\theta_t} & 0 & \dots & 0 \\ 0 & \dots & 0 & -\frac{N_{x_{2t}^1}}{\theta_t} & \dots & -\frac{N_{x_{2t}^{k_2}}}{\theta_t} \\ 0 & \dots & 0 & -\frac{N_{x_{2t}^1}}{\theta_t} & \dots & -\frac{N_{x_{2t}^{k_2}}}{\theta_t} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{N_{x_{2t}^1}}{\theta_t} & \dots & -\frac{N_{x_{2t}^{k_2}}}{\theta_t} \\ 0 & \dots & 0 & \frac{N_{x_{2t}^1}}{\theta_t} & \dots & \frac{N_{x_{2t}^{k_2}}}{\theta_t} \end{bmatrix}$$

as laid out in equation 9 of the paper. Substituting this into the OVB formula given in equation 6 gives the bias on each of the k_1 treatment effects terms β_2 (corresponding to rows $(2+k_1)$ to $(2k_1+1)$ of the δ matrix above) as defined in equation 10 of the paper:

$$E[\widehat{\beta}_2^{k,ovb}|X] \quad = \quad \beta_2^k - \gamma_2^1 \frac{N_{x_{2t}^1}}{\theta_t} - \gamma_2^2 \frac{N_{x_{2t}^2}}{\theta_t} - \gamma_2^{k_2} \frac{N_{x_{2t}^{k_2}}}{\theta_t} \quad \forall k \in 1, \dots, k_1. \tag{A11}$$

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